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Co- and counter-propagating wave effects in an absorbing medium

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ABSTRACT

In this semi-tutorial paper, we revisit the interference phenomena caused by pairs of co-propagating or counter-propagating transverse electromagnetic waves by letting the host medium be absorbing. We first consider plane waves in an unbounded medium, summarize the standing-wave solution of the Maxwell equations, and discuss specific effects caused by nonvanishing absorption. We then consider the superposition of plane and spherical waves in the context of far-field electromagnetic scattering by a particle. To this end we modify the classical Jones lemma by allowing nonzero absorption in the host medium and consider its most obvious consequences such as forward- and backscattering interference. The physical similarity of the two scenarios (superpositions of plane waves and superpositions of plane and spherical waves) is discussed.

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1. Introduction

The existence of remarkable interference phenomena caused by pairs of co-propagating or counter-propagating transverse electromagnetic waves is well known. A classical example is the standing wave resulting from the interference of two counter-propagating plane waves [1–5]. Similarly, the interference of the incident plane wave and the co-propagating far-field spherical wave scattered by a particle in near-forward directions causes the celebrated phenomenon of extinction [6,7].

Traditionally such optical effects have been studied by assuming that the host medium is nonabsorbing [1–19]. Analyzing the case of an absorbing host makes mathematics more involved and can lead to perplexing conclusions which, in the words of Grzesik [20], "any… reasonable person would wish to avoid." Yet we believe that the frequent occurrence of absorbing host media makes such an analysis intellectually instructive as well as practically useful (e.g., [21–24]). Hence the objective of this semi-tutorial paper is to revisit the interference phenomena caused by pairs of co-propagating or counter-propagating transverse electromagnetic waves by letting the host medium be absorbing. Part of our motivation comes from the fact that absorption in the host medium makes nonzero the result of a "standing-wave-like" interaction of the incident plane wave and the counter-propagating spherical wave backscattered by the particle. The latter phenomenon was identified in Refs. [25,26], yet the level of mathematical rigor in those publications was somewhat insufficient and, in our opinion, needs to be refined.

We start by considering the simple cases of co- and counter-propagating plane waves in an unbounded absorbing medium, summarize the standing-wave solution of the macroscopic Maxwell equations, and analyze the most obvious physical consequences of nonzero absorption. We then consider combinations of co- and counter-propagating plane and spherical waves appearing in the context of far-field electromagnetic scattering by a finite object. To this end, we generalize the classical Jones lemma [27,28] to the case of an absorbing host medium and consider its implications such as the forward-scattering (aka extinction) and backscattering interference. In particular, specific consequences of our analysis for the computation of the electromagnetic energy budget of a finite volume of space as well as for the use of laboratory measurements of electromagnetic scattering by a particle are discussed.

2. Maxwell curl equations

Consistent with Refs. [9–18,29–32], in this paper we assume and suppress the $\exp(-\mathrm{i}\omega t)$ time-harmonic dependence of all electromagnetic fields, where $\mathrm{i} = (-1)^{1/2}$, ω is the angular frequency, and t is time. In the absence of impressed currents, the time-independent part of the electromagnetic field everywhere in the three-dimensional linear, isotropic, and nonmagnetic space satis-

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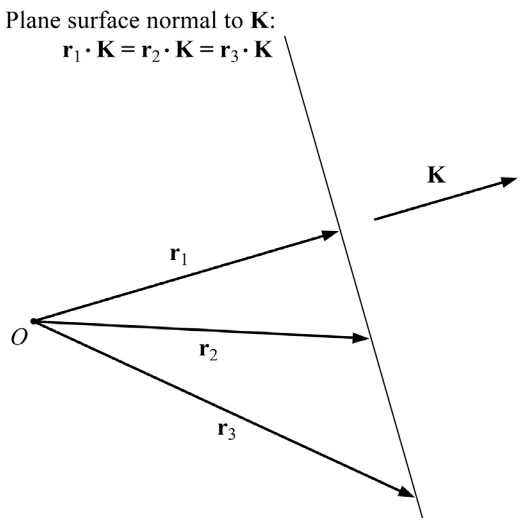

Fig. 1. Plane surface normal to a real vector **K**.

fies the macroscopic Maxwell curl equations

$$\left.\begin{array}{l}\nabla\times\mathbf{E}(\mathbf{r})=\mathrm{i}\omega\mu_0\mathbf{H}(\mathbf{r})\\ \nabla\times\mathbf{H}(\mathbf{r})=-\mathrm{i}\omega\varepsilon(\mathbf{r})\mathbf{E}(\mathbf{r})\end{array}\right\}\mathbf{r}\in\mathbb{R}^3, \tag{1}$$

where the position vector $\mathbf{r}$ connects the origin O of the reference frame and the observation point, $\mathbf{E}(\mathbf{r})$ is the electric and $\mathbf{H}(\mathbf{r})$ the magnetic field; μ_0 is the magnetic permeability of a vacuum; and $\varepsilon(\mathbf{r})$ is the electric permittivity. We will always assume that all materials involved are passive [29,32], which implies that $\varepsilon'' \triangleq \mathrm{Im}[\varepsilon(\mathbf{r})] \geq 0$, where "Im" stands for "imaginary part of".

3. Co-propagating and counter-propagating homogeneous plane waves in a homogeneous unbounded medium

3.1. Homogeneous plane wave

Consider the case of a homogeneous and, in general, absorbing medium. A well-known fundamental solution of Eqs. (1) is a plane electromagnetic wave given by [33]

$$\mathbf{E}(\mathbf{r}) = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathbf{E}_0, \tag{2}$$

$$\mathbf{H}(\mathbf{r}) = \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r})\mathbf{H}_0, \tag{3}$$

where the amplitudes $\mathbf{E}_0$ and $\mathbf{H}_0$ and the wave vector $\mathbf{k}$ are constant complex vectors such that

$$\mathbf{k}\cdot\mathbf{E}_0 = 0, \tag{4}$$

$$\mathbf{k}\cdot\mathbf{H}_0 = 0, \tag{5}$$

$$\mathbf{k}\times\mathbf{E}_0 = \omega\mu_0\mathbf{H}_0, \tag{6}$$

$$\mathbf{k}\times\mathbf{H}_0 = -\omega\varepsilon\mathbf{E}_0. \tag{7}$$

The wave vector is usually written as

$$\mathbf{k} = \mathbf{k}' + \mathrm{i}\mathbf{k}'', \tag{8}$$

where $\mathbf{k}'$ and $\mathbf{k}''$ are real vectors and it is assumed that $\mathbf{k}' \neq \mathbf{0}$. It is easily shown that

$$\mathbf{k}\cdot\mathbf{k} = \omega^2\varepsilon\mu_0. \tag{9}$$

A plane surface normal to a real vector $\mathbf{K}$ satisfies $\mathbf{r}\cdot\mathbf{K} = \mathrm{const}$, where $\mathbf{r}$ is the position vector of a point in the plane (Fig. 1). Therefore, the vector $\mathbf{k}'$ is perpendicular to the surfaces of constant phase, whereas $\mathbf{k}''$ is perpendicular to the surfaces of constant amplitude.

In what follows, we will simplify the problem by considering only homogeneous (or uniform) plane electromagnetic waves defined such that $\mathbf{k}'$ and $\mathbf{k}''$ are parallel (including the case $\mathbf{k}'' = \mathbf{0}$). Then the complex wave vector can be expressed as

$$\mathbf{k} = k\hat{\mathbf{k}} = (k' + \mathrm{i}k'')\hat{\mathbf{k}}, \tag{10}$$

where $\hat{\mathbf{k}}$ is a real unit vector in the direction of propagation and

$$k = \omega\sqrt{\varepsilon\mu_0} \tag{11}$$

is the complex wave number with real and imaginary parts $k' > 0$ and $k'' \geq 0$, respectively.

3.2. Co-propagating plane waves

The sum of two homogeneous-space solutions of the Maxwell equations is also a homogeneous-space solution. Let us first consider two homogeneous plane waves propagating in the same direction $\hat{\mathbf{k}}$, having the same angular frequencies, and represented by the electric field amplitudes $\mathbf{E}_{01}$ and $\mathbf{E}_{02}$ (see Eq. (2)). Obviously, the resulting electromagnetic field is given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) = \exp(\mathrm{i}k\hat{\mathbf{k}}\cdot\mathbf{r})(\mathbf{E}_{01}+\mathbf{E}_{02}), \tag{12}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_1(\mathbf{r}) + \mathbf{H}_2(\mathbf{r}) = \frac{k}{\omega\mu_0}\exp(\mathrm{i}k\hat{\mathbf{k}}\cdot\mathbf{r})\hat{\mathbf{k}}\times(\mathbf{E}_{01}+\mathbf{E}_{02}) \tag{13}$$

and is also a homogeneous plane wave. Thus superposing two co-propagating homogeneous plane waves with the same angular frequencies yields a homogeneous plane wave propagating in the same direction and having the same angular frequency.

The time-averaged Poynting vector of the resulting wave is given by

$$\langle\mathbf{S}(\mathbf{r},t)\rangle_t = \frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})], \tag{14}$$

where "Re" stands for "real part of" and the asterisk denotes complex conjugation. It can be represented as the sum of three terms:

$$\langle\mathbf{S}(\mathbf{r},t)\rangle_t = \mathbf{S}_1(\mathbf{r}) + \mathbf{S}_2(\mathbf{r}) + \mathbf{S}^{\mathrm{int}}(\mathbf{r}), \tag{15}$$

where

$$\mathbf{S}_1(\mathbf{r}) = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_1(\mathbf{r})\times[\mathbf{H}_1(\mathbf{r})]^*\} = \frac{k'}{2\omega\mu_0}\exp(-2k''\hat{\mathbf{k}}\cdot\mathbf{r})|\mathbf{E}_{01}|^2\hat{\mathbf{k}} \tag{16}$$

and

$$\mathbf{S}_2(\mathbf{r}) = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_2(\mathbf{r})\times[\mathbf{H}_2(\mathbf{r})]^*\} = \frac{k'}{2\omega\mu_0}\exp(-2k''\hat{\mathbf{k}}\cdot\mathbf{r})|\mathbf{E}_{02}|^2\hat{\mathbf{k}} \tag{17}$$

are the Poynting vector components associated with the first and second wave taken in isolation from each other, while $\mathbf{S}^{\mathrm{int}}(\mathbf{r})$ is the interference term describing the "interaction" between the two waves and given by

$$\mathbf{S}^{\mathrm{int}}(\mathbf{r}) = \frac{1}{2}\mathrm{Re}\{\mathbf{E}_1(\mathbf{r})\times[\mathbf{H}_2(\mathbf{r})]^* + \mathbf{E}_2(\mathbf{r})\times[\mathbf{H}_1(\mathbf{r})]^*\} = \frac{k'}{\omega\mu_0}\exp(-2k''\hat{\mathbf{k}}\cdot\mathbf{r})\mathrm{Re}(\mathbf{E}_{01}\cdot\mathbf{E}_{02}^*)\hat{\mathbf{k}}. \tag{18}$$

Eqs. (16)–(18) obviously imply that all three components of the Poynting vector

- have the same direction $\hat{\mathbf{k}}$;
- are equally attenuated by the exponential factor $\exp(-2k''\hat{\mathbf{k}}\cdot\mathbf{r})$; and
- survive the limit of zero absorption $k''\to 0$.

Quite predictably, if $\mathbf{E}_{01} = \mathbf{E}_{02}$ then the total Poynting vector is equal to four times that of an individual wave.

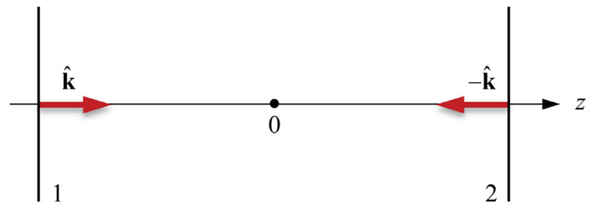

Fig. 2. Two counter-propagating homogeneous plane waves.

3.3. Counter-propagating plane waves

The situation changes substantially if one superposes two counter-propagating plane waves having the same angular frequencies. Assuming that $\hat{\mathbf{k}}$ defines the positive direction of the z-axis, $\hat{\mathbf{k}} = \hat{\mathbf{z}}$ (see Fig. 2), the two counter-propagating homogeneous plane waves, 1 and 2, are given by the following electric and magnetic fields:

$$\mathbf{E}_1(\mathbf{r}) = \exp(ikz)\mathbf{E}_{01}, \tag{19}$$

$$\mathbf{H}_1(\mathbf{r}) = \frac{k}{\omega\mu_0}\exp(ikz)\hat{\mathbf{z}} \times \mathbf{E}_{01}, \tag{20}$$

$$\mathbf{E}_2(\mathbf{r}) = \exp(-ikz)\mathbf{E}_{02}, \tag{21}$$

$$\mathbf{H}_2(\mathbf{r}) = -\frac{k}{\omega\mu_0}\exp(-ikz)\hat{\mathbf{z}} \times \mathbf{E}_{02}. \tag{22}$$

Note that the amplitudes $\mathbf{E}_{01}$ and $\mathbf{E}_{02}$ define the corresponding electric field values at the origin $z = 0$. The resulting total field is given by

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}), \tag{23}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}_1(\mathbf{r}) + \mathbf{H}_2(\mathbf{r}), \tag{24}$$

while the corresponding time-avegared Poynting vector reads

$$\langle \mathbf{S}(\mathbf{r},t)\rangle_t = \frac{1}{2}\mathrm{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})] = \mathbf{S}_1(\mathbf{r}) + \mathbf{S}_2(\mathbf{r}) + \mathbf{S}^{\mathrm{int}}(\mathbf{r}), \tag{25}$$

where

$$\mathbf{S}_1(\mathbf{r}) = \frac{k'}{2\omega\mu_0}\exp(-2k''z)|\mathbf{E}_{01}|^2\hat{\mathbf{z}} \tag{26}$$

and

$$\mathbf{S}_2(\mathbf{r}) = -\frac{k'}{2\omega\mu_0}\exp(2k''z)|\mathbf{E}_{02}|^2\hat{\mathbf{z}} \tag{27}$$

are the Poynting vectors of either wave 1 or wave 2 as if the other wave did not exist, and

$$\mathbf{S}^{\mathrm{int}}(\mathbf{r}) = \frac{k''}{\omega\mu_0}\mathrm{Im}\left[\exp(-2ik'z)\mathbf{E}_{02} \cdot \mathbf{E}_{01}^*\right]\hat{\mathbf{z}} \tag{28}$$

is the interference term.

The divergence of $\langle \mathbf{S}(\mathbf{r},t)\rangle_t$ is given by

$$\begin{aligned}\nabla \cdot \langle \mathbf{S}(\mathbf{r},t)\rangle_t &= -\frac{k'k''}{\omega\mu_0}\exp(-2k''z)|\mathbf{E}_{01}|^2 - \frac{k'k''}{\omega\mu_0}\exp(2k''z)|\mathbf{E}_{02}|^2 \\ &\quad - \frac{2k'k''}{\omega\mu_0}\mathrm{Re}\left[\exp(-2ik'z)\mathbf{E}_{02} \cdot \mathbf{E}_{01}^*\right] \\ &= -\frac{\omega}{2}\varepsilon''|\mathbf{E}(\mathbf{r})|^2.\end{aligned} \tag{29}$$

The last equality is a general fact for a non-magnetic medium, see, e.g., [34]. Since we are dealing with a passive medium, it is easily verified that the localized absorption is always non-negative:

$$-\nabla \cdot \langle \mathbf{S}(\mathbf{r},t)\rangle_t \geq 0. \tag{30}$$

Unlike the relatively simple situation in Section 3.2, the above equations are rich with new physics. To begin with, the first two terms on the right-hand side (RHS) of Eq. (25) describe long-range transport of electromagnetic energy in the form of counter-propagating components. If the host medium is nonabsorbing ($k'' = 0$) then the interference term on the RHS of Eq. (25) vanishes, and we simply have

$$\langle \mathbf{S}(\mathbf{r},t)\rangle_t = \mathbf{S}_1(\mathbf{r})|_{k''=0} + \mathbf{S}_2(\mathbf{r})|_{k''=0}. \tag{31}$$

Furthermore, in the case $|\mathbf{E}_{01}|^2 = |\mathbf{E}_{02}|^2$ we have the classical case of a standing wave transporting no electromagnetic energy whatsoever [3].

The interference term (28) does not describe long-range transport of electromagnetic energy since it oscillates with a period Z equal to half the wavelength,

$$Z = \frac{\lambda}{2}, \tag{32}$$

with

$$\lambda = \frac{2\pi}{k'}, \tag{33}$$

and the amplitude of these oscillations is completely independent of z. According to Eq. (29), the corresponding oscillations in $\nabla \cdot \langle \mathbf{S}(\mathbf{r},t)\rangle_t$ redistribute (modulate) absorption within each Z-long segment of the z-axis. However, the total amount of energy absorbed within each such segment remains unchanged since

$$\int_z^{z+Z} dz \exp(-2ik'z) \equiv 0. \tag{34}$$

An instructive example of Eq. (29) is the case $\mathbf{E}_{01} = \mathbf{E}_{02} = \mathbf{E}_0$. Then the interference term (28) has maxima at $z = N\lambda/2$ ($N = 0, \pm1, \pm2, \ldots$) called the *antinodes* of the total electric field

$$\mathbf{E}(\mathbf{r}) = 2\cos(kz)\mathbf{E}_0, \tag{35}$$

that is at z values corresponding to maximal values of $|\mathbf{E}(\mathbf{r})|^2$. Interestingly (and perhaps not surprisingly), the same situation occurs in the classical demonstration of a standing wave [1] wherein the medium is nonabsorbing, yet the electric field stimulates a photochemical reaction in the otherwise transparent photographic emulsion.

Given the actual physical existence of the interference term in the case $k'' \neq 0$, there are three ways to assess its importance. They can be called local, volumetric, and stochastic. To simplify the analysis, we again assume that $\mathbf{E}_{01} = \mathbf{E}_{02} = \mathbf{E}_0$. First, the independence of the amplitude of the interference term (28) of z suggests that this term can be neglected *locally* whenever

$$\exp(2k''|z|) \gg 1. \tag{36}$$

Depending on the smallness of k'', this condition may or may not be demanding in terms of the requisite z. The same condition implies that $|\mathbf{E}_2(\mathbf{r})| \gg |\mathbf{E}_1(\mathbf{r})|$ or vice versa, so the interference can be neglected along with one of the incoming waves.

To pursue the volumetric approach, let us consider the amount of electromagnetic energy W absorbed per unit time by a cylindrical volume having its rotational axis along the z-axis and its flat bases at $z = -Z_0/2$ and $z = Z_0/2$ (Fig. 3). Obviously,

$$W = S\left[\langle \mathbf{S}(\mathbf{r},t)\rangle_t|_{z=-Z_0/2} - \langle \mathbf{S}(\mathbf{r},t)\rangle_t|_{z=Z_0/2}\right] \cdot \hat{\mathbf{z}}, \tag{37}$$

where S is the area of the flat bases of the cylindrical volume. It is easily seen that the contribution to W of the first two terms on the RHS of Eq. (25) is given by

$$W_{1,2} = S|\mathbf{E}_0|^2\frac{k'}{2\omega\mu_0}\left[\exp(k''Z_0) - \exp(-k''Z_0)\right] \geq 0. \tag{38}$$

It is also clear that the interference term (28) is now equal to

$$\mathbf{S}^{\mathrm{int}}(\mathbf{r}) = -\frac{k''}{\omega\mu_0}\sin(2k'z)|\mathbf{E}_0|^2\hat{\mathbf{z}}, \tag{39}$$

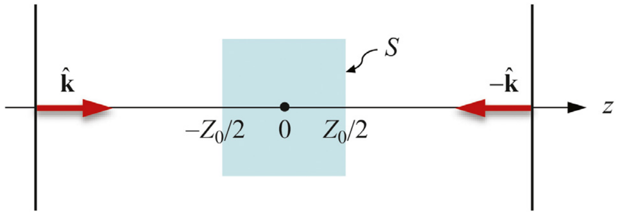

Fig. 3. Electromagnetic energy budget of a cylindrical volume.

and so its contribution to W always satisfies the inequality

$$\left|W^{\text{int}}\right| \le S|\mathbf{E}_0|^2 \frac{2k''}{\omega\mu_0}. \tag{40}$$

Comparison of Eqs. (38) and (40) shows that

$$k'Z_0 \gg 1 \Rightarrow \left|W^{\text{int}}\right| \ll W_{1,2}, \tag{41}$$

irrespective of $k''Z_0$.

Finally, the gist of the stochastic approach is to assume that the boundaries $-Z_0/2$ and $Z_0/2$ of the cylindrical volume in Fig. 3 are not well defined and fixed but rather fluctuate over time randomly with an amplitude of the order of λ. This makes the product $k''(\omega\mu_0)^{-1}|\mathbf{E}_0|^2\sin(2k'z)$ at either base of the cylindrical volume a random variable fluctuating between $-k''(\omega\mu_0)^{-1}|\mathbf{E}_0|^2$ and $+k''(\omega\mu_0)^{-1}|\mathbf{E}_0|^2$, with a long-term average of zero. Obviously, the contribution (38) is virtually immune to this kind of averaging unless absorption in the host medium is extremely strong. Thus the stochastic approach allows one to ignore the interference term on the RHS of Eq. (25) in radiation budget applications involving typical experimental uncertainties.

4. Superposition of plane and spherical waves

Another fundamental solution of the macroscopic Maxwell equations in free space is an incoming or outgoing spherical wave. As was the case with co-propagating plane waves, superposing two incoming or two outgoing spherical waves would result in another incoming or outgoing wave with little new physics. Superposing incoming and outgoing spherical waves is more instructive and can be analyzed along the lines of the preceding section. A more consequential scenario is a superposition of a plane and a spherical wave. A practically important example is the interference of a plane incident wave and the outgoing spherical wave generated in the far field of a finite scattering object. Far-field scattering has been studied extensively in the case of a nonabsorbing host medium [6–20]. However, allowing for nonzero absorption in the host medium results in added mathematical complexity as well as added physics and will therefore be analyzed in this section.

4.1. Far-field scattering by a finite object

Let us consider an impressed [35] homogeneous plane wave given by

$$\mathbf{E}^{\text{inc}}(\mathbf{r}) = \exp(ik\hat{\mathbf{k}}\cdot\mathbf{r})\mathbf{E}_0, \tag{42}$$

$$\mathbf{H}^{\text{inc}}(\mathbf{r}) = \frac{k}{\omega\mu_0}\exp(ik\hat{\mathbf{k}}\cdot\mathbf{r})\hat{\mathbf{k}}\times\mathbf{E}_0, \tag{43}$$

where k is in general complex and "inc" stands for "incident". The presence of a foreign object (a "particle") in the otherwise homogeneous host medium modifies the total electromagnetic field which can now be represented as a superposition of the impressed incident and object-caused scattered ("sca") fields [35]:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + \mathbf{E}^{\text{sca}}(\mathbf{r}), \tag{44}$$

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}^{\text{inc}}(\mathbf{r}) + \mathbf{H}^{\text{sca}}(\mathbf{r}). \tag{45}$$

Let us assume that the particle is centered at the origin O of the reference frame. A fundamental consequence of the so-called radiation condition at infinity [30,36,37] is that at a large distance from the object in its far zone, the scattered field becomes an outgoing transverse spherical wave given by [26,35]

$$\mathbf{E}^{\text{sca}}(\mathbf{r}) = \frac{\exp(ikr)}{r}\mathbf{E}_1(\hat{\mathbf{r}}), \tag{46}$$

$$\mathbf{H}^{\text{sca}}(\mathbf{r}) = \frac{k}{\omega\mu_0}\frac{\exp(ikr)}{r}\hat{\mathbf{r}}\times\mathbf{E}_1(\hat{\mathbf{r}}), \tag{47}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector defining the scattering direction and

$$\hat{\mathbf{r}}\cdot\mathbf{E}_1(\hat{\mathbf{r}}) = 0. \tag{48}$$

Thus the total field becomes the superposition of a plane and a spherical wave:

$$\mathbf{E}(\mathbf{r}) = \exp(ik\hat{\mathbf{k}}\cdot\mathbf{r})\mathbf{E}_0 + \frac{\exp(ikr)}{r}\mathbf{E}_1(\hat{\mathbf{r}}), \tag{49}$$

$$\mathbf{H}(\mathbf{r}) = \frac{k}{\omega\mu_0}\exp(ik\hat{\mathbf{k}}\cdot\mathbf{r})\hat{\mathbf{k}}\times\mathbf{E}_0 + \frac{k}{\omega\mu_0}\frac{\exp(ikr)}{r}\hat{\mathbf{r}}\times\mathbf{E}_1(\hat{\mathbf{r}}). \tag{50}$$

Note that the scattered spherical wavefront and the incident plane wave are co-propagating in forward-scattering directions, i.e., when $\hat{\mathbf{r}}$ is equal or very close to $\hat{\mathbf{k}}$, and counter-propagating at backscattering directions, i.e., when $\hat{\mathbf{r}}$ is equal or very close to $-\hat{\mathbf{k}}$.

If the amplitude $\mathbf{E}_1(\hat{\mathbf{r}})$ can be found as a computer solution of the macroscopic Maxwell equations (e.g., Ref. [38–40]) then the formulas (49) and (50) are self-sufficient in that they enable the numerical calculation of any far-field optical observable in the form of a second moment in the field. Yet it is instructive to do as much work analytically as possible. This is especially true of the forward-scattering and backscattering directions in the far zone since then we have pairs of plane and quasi-plane wavefronts causing extinction and quasi-standing wave effects, respectively. To analyze these effects we will need a generalization of the Jones lemma (otherwise known as the Saxon decomposition) [27,28] to the case of an absorbing host medium. We will discuss this generalization in the following subsection.

4.2. Generalized Jones lemma

The classical Jones lemma is typically formulated as follows [3]:

$$\oint_{4\pi} d\hat{\mathbf{r}}\, f(\hat{\mathbf{r}})\exp(ia\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) \underset{a\to\infty}{\to} \frac{2\pi}{ia}[f(\hat{\mathbf{k}})\exp(ia) - f(-\hat{\mathbf{k}})\exp(-ia)], \tag{51}$$

where $f(\hat{\mathbf{r}})$ is a sufficiently smooth function (at least, has a continuous second derivative), the integration is performed over the unit sphere of outgoing directions $\hat{\mathbf{r}}$, and a is real and positive. In the far-field scattering problem formulated in the preceding subsection, $a = kr$. The proof of the Jones lemma in Ref. [3] is rather elementary but employs a two-dimensional form of the method of stationary phase. That approach was probably inherited from the original discussion by Jones [27] wherein the lemma is formulated in a more general setting by allowing the integration domain to cover only part of the unit sphere. This makes certain spherical coordinate systems for $\hat{\mathbf{r}}$ preferable over the others. However, such a proof is needlessly complicated in application to the integral (51).

Let us define the spherical coordinate system (r, θ, φ) such that the direction $\theta = 0$ is along $\hat{\mathbf{k}}$. This is a critical simplification afforded by the symmetry of the integration domain. Then

$$\oint_{4\pi} \mathrm{d}\hat{\mathbf{r}} f(\hat{\mathbf{r}}) \exp(\mathrm{i}a\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) = \int_0^\pi \mathrm{d}\theta \sin\theta \exp(\mathrm{i}a\cos\theta) \int_0^{2\pi} \mathrm{d}\varphi f(\theta,\varphi) = \int_{-1}^1 \mathrm{d}\eta g(\eta) \exp(\mathrm{i}a\eta), \quad (52)$$

where $\eta \triangleq \cos\theta$ and

$$g(\eta) \triangleq h(\arccos\eta),\ h(\theta) \triangleq \int_0^{2\pi} \mathrm{d}\varphi f(\theta,\varphi). \quad (53)$$

Evidently, the smoothness of $h(\theta)$ is not worse than that of $f(\theta, \varphi)$, and so is the smoothness of $g(\eta)$ for $|\eta| < 1$ since

$$g'(\eta) = -\frac{h'(\arccos\eta)}{\sqrt{1-\eta^2}}. \quad (54)$$

The points $\eta = \pm 1$ need special consideration. Let us extend the range of θ to negative values according to $f(-\theta,\varphi) \triangleq f(\theta, 2\pi - \varphi)$ and further periodically with a period of 2π. This extension does not affect the smoothness of f, i.e., f is smooth for any real θ. Then Eq. (53) implies that $h(-\theta) \equiv h(\theta)$ and $h(\pi - \theta) \equiv h(\pi + \theta)$, i.e., h is symmetric with respect to both $\theta = 0$ and $\theta = \pi$. Hence, all existing odd-order derivatives of h vanish at these special points. In particular, using the L'Hôpital's rule we obtain

$$g'(1) = -h''(0),\ g'(-1) = h''(\pi). \quad (55)$$

We can extend this analysis to higher derivatives, but we skip immediately to the most practically relevant application, that is, infinitely smooth functions $f(\hat{\mathbf{r}})$ and $h(\theta)$ having continuous derivatives of any order. Then $h(\theta)$ can be expanded into a Fourier series that, owing to the symmetry properties, includes only cosines:

$$h(\theta) = \sum_{n=0}^{\infty} h_n \cos(n\theta) = \sum_{n=0}^{\infty} h_n T_n(\cos\theta), \quad (56)$$

where T_n are Chebyshev polynomials [41]. Moreover, Eq. (56) *uniformly* approximates $h(\theta)$ and its derivatives of any order. Hence, the values of $g(\eta)$ and its derivatives of any order at $\eta = \pm 1$ can be obtained from this series and are finite. The same result can be obtained by expanding $f(\theta, \varphi)$ into spherical harmonics (see, e.g., Ref. [9]) which are expressed in Legendre polynomials of $\cos\theta$. In other words, η is overall a more natural variable than θ for the description of smooth functions on the unit sphere. Instead of special points (poles) for θ we have simple interval endpoints for η.

Going back to Eq. (52), the asymptotic expansion can be obtained via integration by parts, which is a powerful approach to obtain such expansions [9,20,25]:

$$\int_{-1}^1 \mathrm{d}\eta g(\eta) \exp(\mathrm{i}a\eta) = \left.\frac{g(\eta)\exp(\mathrm{i}a\eta)}{\mathrm{i}a}\right|_{-1}^1 - \left.\frac{g'(\eta)\exp(\mathrm{i}a\eta)}{(\mathrm{i}a)^2}\right|_{-1}^1 + \frac{1}{(\mathrm{i}a)^2}\int_{-1}^1 \mathrm{d}\eta g''(\eta)\exp(\mathrm{i}a\eta) = \frac{1}{\mathrm{i}a}[g(1)\exp(\mathrm{i}a) - g(-1)\exp(-\mathrm{i}a)] + O(a^{-2}), \quad (57)$$

where the remaining integral is bounded (independently of a) since

$$\left|\int_{-1}^1 \mathrm{d}\eta g''(\eta)\exp(\mathrm{i}a\eta)\right| \le \int_{-1}^1 \mathrm{d}\eta \left|g''(\eta)\right|, \quad (58)$$

while the continuity of $h''(\theta)$ ensures integrability of $|g''(\eta)|$ even if the latter diverges at $\eta = \pm 1$. Recalling that $g(\pm 1) = 2\pi f(\pm\hat{\mathbf{k}})$, one can see that Eq. (57) is exactly the Jones lemma (51), but with an explicit remainder.

For infinitely smooth $g(\eta)$ we can easily obtain an asymptotic series of arbitrary order N via further integration by parts:

$$\int_{-1}^1 \mathrm{d}\eta g(\eta)\exp(\mathrm{i}a\eta) = \frac{1}{\mathrm{i}a}\sum_{n=0}^{N}\frac{1}{(\mathrm{i}a)^n}[g^{(n)}(1)\exp(\mathrm{i}a) - g^{(n)}(-1)\exp(-\mathrm{i}a)] + O(a^{-N-2}), \quad (59)$$

where $g^{(n)}$ denotes the n-th derivative of g. This result would be hard to obtain without the transformation (52).

Let us now extend the Jones lemma to a complex $a \triangleq a' + \mathrm{i}a''$. All the above formalism remains valid with the exception of the estimate of Eq. (58) which should be modified into

$$\left|\int_{-1}^1 \mathrm{d}\eta g''(\eta)\exp(\mathrm{i}a\eta)\right| \le \exp\left|a''\right| \int_{-1}^1 \mathrm{d}\eta\left|g''(\eta)\right|. \quad (60)$$

The generalized Jones lemma is then

$$\oint_{4\pi} \mathrm{d}\hat{\mathbf{r}} f(\hat{\mathbf{r}})\exp(\mathrm{i}a\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) = \int_{-1}^1 \mathrm{d}\eta g(\eta)\exp(\mathrm{i}a\eta) = \frac{\exp(\mathrm{i}a)}{\mathrm{i}a}\left[g(1) + O(a^{-1})\right] - \frac{\exp(-\mathrm{i}a)}{\mathrm{i}a}\left[g(-1) + O(a^{-1})\right], \quad (61)$$

or, including the higher orders,

$$\int_{-1}^1 \mathrm{d}\eta g(\eta)\exp(\mathrm{i}a\eta) = \frac{\exp(\mathrm{i}a)}{\mathrm{i}a}\left[\sum_{n=0}^{N}\frac{g^{(n)}(1)}{(\mathrm{i}a)^n} + O(a^{-N-1})\right] - \frac{\exp(-\mathrm{i}a)}{\mathrm{i}a}\left[\sum_{n=0}^{N}\frac{g^{(n)}(-1)}{(\mathrm{i}a)^n} + O(a^{-N-1})\right]. \quad (62)$$

While this result looks very similar to that for a real a, it has an important new feature. Strictly speaking, we have obtained a *compound* asymptotic expansion [42]. The major problem then is that the remainder from one part on the RHS of Eq. (61) can be larger than the main term of the other part, the factors $\exp(\pm\mathrm{i}a)$ serving as the principal disruptor. For example, even if $g(-1) = 0$, one still cannot conclude that the term with $g(1)$ in Eq. (61) will provide the largest contribution in the case of a large positive a''. Thus, while being rigorous and applicable to any complex a, this asymptotic expansion is of limited use.

Fortunately, not all values of a are relevant to practical applications. Going back to the scattering problem, a complex a corresponds to a complex k, and k'' is responsible for absorption in the host medium. If absorption is relatively strong then one does not receive any measurable signal at a very distant point in the first place. To have any chance of defining meaningful far-field optical observables, we must assume a small k'' (to be specified below). Moreover, taking the far-field limit entails ignoring the terms that are $O(a^{-1})$ relative to the main terms. Thus, we are restricting ourselves to the first-order expansion, Eq. (61), and hence want to ensure the smallness of both remainders.

If $a'' \ge 0$ (corresponding to the passive medium) then the second remainder in Eq. (61) is the larger one, yet we want it to be much smaller than the first part of the expansion. The specific expression for the remainder can be obtained from Eq. (62) leading to the condition

$$\frac{\exp(2a'')}{|a|} \ll \left|\frac{g(1)}{g'(-1)}\right| = \left|\frac{h(0)}{h''(\pi)}\right| \triangleq \frac{1}{\xi_\pi}, \quad (63)$$

where we used Eq. (55). The most important consequence is that this condition *always* fails for a large enough $|a|$ if $\arg(a) \ne 0$. This makes it impossible to perform a simple analysis based on a "large enough" argument. However, Eq. (63) may still be valid in the numerical sense for a wide range of $|a|$. Using the fact that $x = \ln\beta$ is an approximate solution (within a factor of 1.6) to $\exp x = \beta x$ for $\beta > e$ and assuming that $0 \le \arg(a) \ll 1$, we can rewrite Eq. (63) as follows:

$$|a| \ll -\frac{\ln[2\xi_\pi \arg(a)]}{2\arg(a)}. \tag{64}$$

Actually, Eq. (64) is a more stringent condition due to the sharp increase of the left-hand side of Eq. (63) near this boundary value. More accurate bounds can be obtained by using $x = \ln(\beta\ln\beta)$ instead of $x = \ln\beta$.

On the lower side of the range of $|a|$, we need $|a| \gg \xi_\pi$, but considering the smallness of the remainder at $\theta = 0$ we additionally require $|a| \gg \xi_0$, where

$$\xi_0 \triangleq \left|\frac{h''(0)}{h(0)}\right|. \tag{65}$$

For scattering problems with $f(\hat{\mathbf{r}})$ being a scattering amplitude, typically $\xi_0 \ge \xi_\pi$ and $\xi_0 \sim [\max(|k|D, 1)]^2$, where D is the largest particle dimension. Thus, the condition $|a| \gg \xi_0$ describes the well-known far-field limit: $|k|r \gg 1$ and $r \gg |k|D^2$ [26].

Still, not to restrict our discussion, we define $\xi \triangleq \max(\xi_0, \xi_\pi)$ and obtain the final condition for the validity of the generalized Jones lemma:

$$\xi \ll |a| \ll -\frac{\ln[2\xi_\pi \arg(a)]}{2\arg(a)}, \tag{66}$$

where the latter asymptotic relation is due to the fact that $|\ln(2\xi_\pi)|$ is much smaller than the typical $|a|$ considered in the context of the far-field limit (at least $\gtrsim$100). In this range of $|a|$ we can employ Eq. (51) as is, with the only caveat that instead of the asymptotic condition with arbitrary accuracy for a large enough real a we have an approximation with "good" numerical accuracy in the specified range of $|a|$. The specific estimates for the remainder are discussed above.

Finally, let us discuss the required smallness of absorption. Eq. (66) implies that

$$2\xi\arg(a) \ll \ln\left(\frac{\xi}{\xi_\pi}\right), \tag{67}$$

where "$\ll$" should mean at least several orders of magnitude to make a reasonably useful range of $|a|$ in Eq. (66). The RHS can be significant for large scatterers, but that is necessarily accompanied by a large ξ. Hence, in all cases we have $\arg(a) \ll 1$, in agreement with the previous assumptions.

Under these conditions we can replace a by a' in the denominator of the RHS of Eq. (51). Thus, we have rigorously substantiated the approximate formula that was "naively" derived in Ref. [26] by applying the original Jones lemma to a function which "weakly" depends on a through the factor $\exp(-a''\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})$:

$$\begin{aligned}\oint_{4\pi} \mathrm{d}\hat{\mathbf{r}} f(\hat{\mathbf{r}})\exp(\mathrm{i}a\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) &= \oint_{4\pi} \mathrm{d}\hat{\mathbf{r}} f(\hat{\mathbf{r}})\exp(-a''\hat{\mathbf{r}}\cdot\hat{\mathbf{k}})\exp(\mathrm{i}a'\hat{\mathbf{r}}\cdot\hat{\mathbf{k}}) \\ &\approx \frac{2\pi}{\mathrm{i}a'}[f(\hat{\mathbf{k}})\exp(-a'')\exp(\mathrm{i}a') \\ &\quad - f(-\hat{\mathbf{k}})\exp(a'')\exp(-\mathrm{i}a')] \\ &= \frac{2\pi}{\mathrm{i}a'}[f(\hat{\mathbf{k}})\exp(\mathrm{i}a) - f(-\hat{\mathbf{k}})\exp(-\mathrm{i}a)],\end{aligned} \tag{68}$$

where "$\approx$" denotes that the result is no longer asymptotically correct in the limit $|a| \to \infty$. Formally Eq. (68) is equivalent to the generalized Saxon's decomposition of a plane wave into an incoming and outgoing spherical waves [26]:

$$\exp(\mathrm{i}a\hat{\mathbf{k}}\cdot\hat{\mathbf{r}}) \approx \frac{\mathrm{i}2\pi}{a'}[\delta(\hat{\mathbf{k}}+\hat{\mathbf{r}})\exp(-\mathrm{i}a) - \delta(\hat{\mathbf{k}}-\hat{\mathbf{r}})\exp(\mathrm{i}a)], \tag{69}$$

where $\delta(\hat{\mathbf{r}})$ is the solid-angle delta function.

4.3. Energy budget of a spherical volume centered at the scattering object

To discuss analytical implications of Sections 4.1 and 4.2, let us evaluate the electromagnetic energy budget of a spherical volume V centered at the scattering object. We assume that the radius r of this volume is in the far zone of the scatterer yet satisfies the criteria of applicability of the approximate Jones lemma (68), i.e.

$$\exp(2k''r) \ll k'r, \tag{70}$$

which is a simplification of Eq. (63) assuming $\xi_\pi \sim 1$. The time-averaged Poynting vector at the boundary S of V can be represented as the sum of three terms:

$$\langle\mathbf{S}(\mathbf{r},t)\rangle_t = \mathbf{S}^{\mathrm{inc}}(\mathbf{r}) + \mathbf{S}^{\mathrm{sca}}(\mathbf{r}) + \mathbf{S}^{\mathrm{int}}(\mathbf{r}), \mathbf{r} \in S, \tag{71}$$

where

$$\begin{aligned}\mathbf{S}^{\mathrm{inc}}(\mathbf{r}) &= \frac{1}{2}\mathrm{Re}\{\mathbf{E}^{\mathrm{inc}}(\mathbf{r}) \times [\mathbf{H}^{\mathrm{inc}}(\mathbf{r})]^*\} \\ &= \frac{k'}{2\omega\mu_0}\exp(-2k''\hat{\mathbf{k}}\cdot\mathbf{r})|\mathbf{E}_0|^2\hat{\mathbf{k}}\end{aligned} \tag{72}$$

and

$$\begin{aligned}\mathbf{S}^{\mathrm{sca}}(\mathbf{r}) &= \frac{1}{2}\mathrm{Re}\{\mathbf{E}^{\mathrm{sca}}(\mathbf{r}) \times [\mathbf{H}^{\mathrm{sca}}(\mathbf{r})]^*\} \\ &= \frac{k'}{2\omega\mu_0}\frac{\exp(-2k''r)}{r^2}\left|\mathbf{E}_1(\hat{\mathbf{r}})\right|^2\hat{\mathbf{r}}\end{aligned} \tag{73}$$

are the Poynting vector components associated with the incident and the scattered field, respectively, while the interference term

$$\mathbf{S}^{\mathrm{int}}(\mathbf{r}) = \frac{1}{2}\mathrm{Re}\{\mathbf{E}^{\mathrm{inc}}(\mathbf{r}) \times [\mathbf{H}^{\mathrm{sca}}(\mathbf{r})]^* + \mathbf{E}^{\mathrm{sca}}(\mathbf{r}) \times [\mathbf{H}^{\mathrm{inc}}(\mathbf{r})]^*\} \tag{74}$$

describes the "interaction" between the incident and scattered fields.

The net time-averaged flow of electromagnetic power entering V (i.e., being absorbed inside V) is given by

$$W^{\mathrm{abs}} = -\oiint_S \mathrm{d}^2\mathbf{r}\langle\mathbf{S}(\mathbf{r},t)\rangle_t\cdot\hat{\mathbf{r}} = -r^2\oint_{4\pi}\mathrm{d}\hat{\mathbf{r}}\langle\mathbf{S}(r\hat{\mathbf{r}},t)\rangle_t\cdot\hat{\mathbf{r}}. \tag{75}$$

According to Eq. (71), W^{abs} can be written as a combination of three terms [43]:

$$W^{\mathrm{abs}} = W^{\mathrm{inc}} - W^{\mathrm{sca}} + W^{\mathrm{int}}, \tag{76}$$

where

$$W^{\mathrm{inc}} = -r^2\oint_{4\pi}\mathrm{d}\hat{\mathbf{r}}\,\mathbf{S}^{\mathrm{inc}}(r\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}, \tag{77}$$

$$W^{\mathrm{sca}} = r^2\oint_{4\pi}\mathrm{d}\hat{\mathbf{r}}\,\mathbf{S}^{\mathrm{sca}}(r\hat{\mathbf{r}})\cdot\hat{\mathbf{r}} = \frac{k'\exp(-2k''r)}{2\omega\mu_0}\oint_{4\pi}\mathrm{d}\hat{\mathbf{r}}|\mathbf{E}_1(\hat{\mathbf{r}})|^2, \tag{78}$$

and

$$W^{\mathrm{int}} = -r^2\oint_{4\pi}\mathrm{d}\hat{\mathbf{r}}\,\mathbf{S}^{\mathrm{int}}(r\hat{\mathbf{r}})\cdot\hat{\mathbf{r}}. \tag{79}$$

To compute W^{inc} and W^{int}, we again direct the z-axis of a spherical coordinate system along $\hat{\mathbf{k}}$. Then

$$\begin{aligned}W^{\mathrm{inc}} &= -\frac{\pi k'r^2}{\omega\mu_0}|\mathbf{E}_0|^2\int_{-1}^{1}\mathrm{d}\eta\,\eta\exp(-2k''r\eta) \\ &= \frac{\pi k'r}{\omega\mu_0 k''}\left[\cosh(2k''r) - \frac{\sinh(2k''r)}{2k''r}\right]|\mathbf{E}_0|^2,\end{aligned} \tag{80}$$

where, as before, $\eta = \cos\theta$. Also, tedious algebra and application of the generalized Jones lemma [Eq. (61)] yields

$$
\begin{aligned}
W^{\mathrm{int}} &= -\frac{1}{2\omega\mu_0}\oint_{4\pi} \mathrm{d}\hat{\mathbf{r}}\,\mathrm{Re}\Big[k^*\Big(\mathbf{E}_0\cdot\mathbf{E}_1^*(\hat{\mathbf{r}})\exp[\mathrm{i}r(k\eta-k^*)] \\
&\quad +\exp[\mathrm{i}r(k-k^*\eta)]\{\eta[\mathbf{E}_0^*\cdot\mathbf{E}_1(\hat{\mathbf{r}})]-(\hat{\mathbf{r}}\cdot\mathbf{E}_0^*)[\hat{\mathbf{k}}\cdot\mathbf{E}_1(\hat{\mathbf{r}})]\}\Big)\Big] \\
&\approx -\frac{\pi}{\omega\mu_0}\mathrm{Im}\Big\{\exp(-2k''r)\Big[\mathbf{E}_0\cdot\mathbf{E}_1^*(\hat{\mathbf{k}})\frac{k^*}{k}-\mathbf{E}_0^*\cdot\mathbf{E}_1(\hat{\mathbf{k}})\Big] \\
&\quad -\exp(-2\mathrm{i}k'r)\mathbf{E}_0\cdot\mathbf{E}_1^*(-\hat{\mathbf{k}})\frac{k^*}{k}-\exp(2\mathrm{i}k'r)\mathbf{E}_0^*\cdot\mathbf{E}_1(-\hat{\mathbf{k}})\Big\} \\
&= W^{\mathrm{ext}}+W^{\mathrm{bi}}, \qquad (81)
\end{aligned}
$$

where

$$
\begin{aligned}
W^{\mathrm{ext}} &\approx \frac{2\pi}{\omega\mu_0}\exp(-2k''r)\Big\{\mathrm{Im}[\mathbf{E}_1(\hat{\mathbf{k}})\cdot\mathbf{E}_0^*]+\frac{k''}{k'}\mathrm{Re}[\mathbf{E}_1(\hat{\mathbf{k}})\cdot\mathbf{E}_0^*]\Big\} \\
&\approx \frac{2\pi}{\omega\mu_0}\exp(-2k''r)\mathrm{Im}[\mathbf{E}_1(\hat{\mathbf{k}})\cdot\mathbf{E}_0^*] \qquad (82)
\end{aligned}
$$

is the "extinction" term (forward-scattering interference) and

$$
\begin{aligned}
W^{\mathrm{bi}} &\approx \frac{2\pi}{\omega\mu_0}\frac{k''}{k'}\Big\{\frac{k''}{k'}\mathrm{Im}[\exp(2\mathrm{i}k'r)\mathbf{E}_1(-\hat{\mathbf{k}})\cdot\mathbf{E}_0^*] \\
&\quad -\mathrm{Re}[\exp(2\mathrm{i}k'r)\mathbf{E}_1(-\hat{\mathbf{k}})\cdot\mathbf{E}_0^*]\Big\} \\
&\approx -\frac{2\pi}{\omega\mu_0}\frac{k''}{k'}\mathrm{Re}[\exp(2\mathrm{i}k'r)\mathbf{E}_1(-\hat{\mathbf{k}})\cdot\mathbf{E}_0^*] \qquad (83)
\end{aligned}
$$

is the "backscattering interference" term. In Eqs. (82) and (83) the first $\approx$ uses only the smallness of k'' implying $k^*/k \approx 1-2\mathrm{i}k''/k'-2(k''/k')^2$ [cf. Eq. (68)], while the second $\approx$ assumes comparable magnitude of the corresponding real and imaginary parts. The latter assumption is actually questionable. For W^{bi} the phase of the corresponding complex value depends on r, thus either imaginary or real parts may vanish – then the assumption is valid only in the stochastic sense discussed below. For W^{ext} the phase of the corresponding complex value is large for significantly absorbing or large scatterers. As a counterexample, let us consider the simplest case of a Rayleigh sphere of radius ρ such that $k'\rho \ll 1$, for which [6]

$$
\mathbf{E}_1(\hat{\mathbf{k}}) = k^2\rho^3\frac{m^2-1}{m^2+2}\mathbf{E}_0, \qquad (84)
$$

where m is the relative refractive index. Here the radiative correction of $O(k^5\rho^6)$ is neglected, although it may become significant for very weak absorption (then W^{sca} is comparable to W^{abs}). If we take a purely real m, the second summand in curly brackets in Eq. (82) is exactly half of the first term and cannot be neglected. To finish the introduction of W^{inc} and W^{bi}, we stress once again that the whole separation of W^{int} into two parts is possible only when Eq. (70) is valid.

It is easily verified that in the limit $k'' \to 0$, W^{inc} and W^{bi} vanish, and we thereby recover the classical results of the scattering theory valid for a nonabsorbing host medium [6,12,17]. Furthermore, the energy budget of a large ("far-field") volume containing the particle becomes independent of the size and shape of the volume, which allows one to define the classical scattering, extinction, and absorption cross sections.

If $k'' > 0$ then both W^{inc} and W^{bi} provide nonzero contributions to W^{abs}. If $k''/k' \ll \exp(-2k''r)$ then W^{bi} can be neglected in comparison with W^{ext}. This condition is commonly satisfied whenever Eq. (70) holds, since

$$
\exp(2k''r) \ll k'r \Rightarrow \exp[2k''(r/2)] \ll \frac{k'r}{\exp(k''r)} \le \frac{k'}{k''}. \qquad (85)
$$

In other words, the suitable range of r satisfying both far-field conditions and Eq. (70), which may span a few orders of magnitude, is reduced by not more than a factor of two. Still, for r at the right boundary of applicability of Eq. (70) W^{bi} can become comparable to and even exceed W^{ext}. But then, by analogy with the stochastic approach outlined in Section 3.3, we can exploit the fact that W^{bi} depends on r only via the complex exponential factor $\exp(\mathrm{i}2k'r)$. Then assuming that r fluctuates randomly with an uncertainty range of about λ makes W^{bi} a random variable with a zero average. Of course, the same outcome would result from the assumption that the scattering particle wobbles randomly around O. Again, all other contributions to W^{abs} are virtually immune to this "stochastization" procedure.

As to the contribution associated with the incident wave, we have in the case $k''r \ll 1$

$$
W^{\mathrm{inc}} = \frac{4\pi}{3}r^3\frac{k'k''}{\omega\mu_0}|\mathbf{E}_0|^2\{1+O[(k''r)^2]\}, \qquad (86)
$$

which can potentially be much smaller than W^{ext}. In the opposite limit $k''r \gg 1$,

$$
W^{\mathrm{inc}} \approx \frac{\pi k'r}{2\omega\mu_0 k''}\exp(2k''r)|\mathbf{E}_0|^2 \qquad (87)
$$

and inevitably dwarfs W^{sca} and W^{int} in Eq. (76), thereby making all consequences of electromagnetic scattering essentially unobservable. More specifically, already at $k''r = 1/2$

$$
\frac{W^{\mathrm{ext}}}{W^{\mathrm{inc}}} = \frac{1}{4r^2}C_{\mathrm{ext}}^{(0)} = O\left(\frac{\max(kD,1)}{kr}\right)^2, \qquad (88)
$$

i.e., the ratio is limited by a square of the value that is assumed $\ll 1$ according to the far-field limit. Here we used the standard definition of the extinction cross section for a non-absorbing medium [12,17]

$$
C_{\mathrm{ext}}^{(0)} = \frac{4\pi}{k'}\frac{\mathrm{Im}[\mathbf{E}_1(\hat{\mathbf{k}})\cdot\mathbf{E}_0^*]}{|\mathbf{E}_0|^2} \qquad (89)
$$

and the fact that it is proportional to the geometrical cross section for particles with the largest dimension $D > \lambda$ or is limited by $O(\lambda^2)$ otherwise (the particle absorption only weakly affects $C_{\mathrm{ext}}^{(0)}$ since $k''D \ll 1$ [44]).

Potential non-negligibility of W^{bi} implies more fundamental limitations on the measurability of the radiation budget of the volume V. Indeed, placing a detector of light such that it faces the backscattered wavefront can effectively block the incident wavefront and thereby destroy the backscattering interference contribution altogether. While measuring $|\mathbf{E}_1(-\hat{\mathbf{k}})|^2$ is possible (at least, close to the exact backscattering direction), it is not sufficient to infer W^{bi} due to the unknown phase. However, even if the above-mentioned stochastic approach is not invoked, W^{bi} is only significant for large $k''r$, when the whole measurement is dominated by W^{inc} [Eq. (87)] making it impossible to discriminate other contributions in view of realistic experimental errors.

5. Discussion

There are obvious analogies between the results of Sections 3 and 4. Indeed, at a large distance from the particle, the scattered field becomes an outgoing spherical wave with a locally quasi-plane wavefront. As a consequence, W^{ext} can be viewed as describing the interference of the incident and co-propagating scattered waves, whereas W^{bi} can be interpreted as quantifying the interference of the incident and counter-propagating scattered waves. It is then not surprising that Eqs. (18) and (82) are similar and describe contributions that decrease exponentially with r and survive the limit $k'' \to 0$. Eqs. (28) and (83) are similar as well and describe contributions that vanish in a nonabsorbing host medium, are oscillating functions of r, and can be eliminated by applying the stochastization procedure.

Despite these similarities, it is imperative to remember that the results of Section 4 are based on the generalization of the Jones lemma that is not asymptotically valid in the limit $|k|r \to \infty$. Therefore, it is the substitution of Eqs. (49) and (50) in the primordial definition (14) coupled with Eq. (75) that needs to be used in order to evaluate numerically the energy budget of the volume V bounded by the far-field surface S. Even if the generalized Jones lemma were asymptotically valid, Eq. (75) could not be used to define the scattering and absorption cross sections in any meaningful way as quantities independent of the size and shape of the enclosing volume V. Even the incident-field contribution to the total energy budget must be explicitly computed for the given volume geometry. The case of a far-field volume not encompassing the particle is even more revealing. Indeed, in the case of a nonabsorbing host medium its electromagnetic energy budget must be independent of the presence of the scattering object and trivial (i.e., identically equal to zero). Yet if $k'' > 0$, it becomes explicitly dependent on the volume's geometry, on the scattering object, and on their relative configuration with respect to the incident wave. There is still hope that under some conditions the scatterer and volume properties can be partially decoupled, as is seen in Eqs. (78) and (82). We leave such an analysis in application to realistic measurement configurations for further research.

Finally, we note that W^{ext} defined by Eq. (82) is a directly measurable quantity using the experimental configuration discussed in Section 13.1 of Ref. [17]. However, it requires the radiometer to be large enough and not perfectly circular to average out the oscillations of the extinction field over its entrance area [45]. Moreover, the rigorous mathematical analysis of such measurements in an absorbing medium has not yet been performed, although we do not foresee any principal differences if the absorption is sufficiently small (as is assumed in the derivation of W^{ext} itself).

Declaration of Competing Interest

None.

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